HOMOLOGICAL ALGEBRA OF HILBERT SPACES ENDOWED WITH A COMPLETE NEVANLINNA-PICK KERNEL

By

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This work is dedicated to the memory of my father Robert James Clancy, who provided me an unassailable example of what a man should strive to be.

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PREFACE

Homological algebra has long been established as a field of research separate from the topological problems of the late nineteenth and early twentieth century which spawned the subject. Indeed, by the middle of this century a rich body of knowledge had been developed and cast in the abstract setting of category theory. Concurrent with the abstraction of these results from the topological setting was a diversification in their application. Although sometimes ridiculed for the level of abstraction, category theory thrived as new and exciting homological applications appeared in group theory, lie algebras, and logic, among other areas as well. Still, the geometric insight afforded by the original toplogical problems remains a powerful influence. Each of the areas mentioned has been cross fertilized by interactions with the other areas by the ability of category theory to give a precision to "analogous" results in different areas of research.

In the following work operator theoretic results which we establish are given homological meaning. The homological framework that has been developed serves not only to provide what we believe to be the proper perspective from which to view these operator theoretic results, but enriches the operator theory by suggesting directions for further research. It is our opinion that when the results which homological algebra seek are established, they will provide fruitful insight into operator theory itself.

It would be unfair however to represent the development of the following results as a strict application of homological algebra to operator theory. As each

area of mathematics develops, there arises an organization of the subject providing some implicit valuation to certain results above others. The organization of the material takes place both by the logical ordering of the work, and also subjectively by the community of mathematicians working in the area. One result in operator theory which has achieved some status in the latter regard is the commmutant lifting theorem. Operator theorists have been very successful in employing the commutant lifting theorem to solve many problems within their discipline. Hence the proof of a generalization of the commutant lifting theorem in our setting provides an operator theoretic rationale for this approach. Additionally, specialists working in closely related areas, e.g. control theory, are taking more and more abstract approaches to difficult problems in their disciplines.

In short, our opinion is that both operator theoretical and homological perspectives are necessary. The interplay between the two is rich and similar relationships have proven to be very powerful in other areas of mathematics.

Our notation is standard for the most part. The field of complex numbers is denoted by $\mathbb{C}$. We use the math fraktur font $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}, \ldots$ to denote categories. Hilbert spaces are always complex and separable and usually written in math script $\mathcal{H}$, or calligraphy $\mathcal{H}$. The set of bounded linear maps between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is written $\mathcal{L}(\mathcal{H}, \mathcal{K})$ or $\mathcal{L}(\mathcal{H})$ if $\mathcal{H} = \mathcal{K}$. Elements of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ are referred to as operators; in particular, operators are bounded. Roman majuscules T, V, $W, \ldots$ will typically be used to denote operators. An important exception to this convention is the model operator $\mathcal{S}_k$ defined in Section 3.6. The definition of a Nevanlinna-Pick reproducing kernel is given in Chapter 3, after which we reserve k to denote a (fixed) Nevanlinna-Pick kernel and refer to k as an NP kernel. Given a Hilbert space $\mathcal{H}$ and an element $h \in \mathcal{H}$, the function $p_h(T) = \|Th\|$ defines a seminorm on $\mathcal{L}(\mathcal{H})$. The topology induced upon $\mathcal{L}(\mathcal{H})$ by the family of seminorms $\{p_h \,|\, h \in \mathcal{H}\}$ is called the *strong operator topology.*

We assume the reader is familiar with the standard results of functional analysis, such as is covered in Conway [9]. Specifically such results as the Banach-Steinhaus theorem, the principle of uniform boundedness, and various convergence criteria in the strong operator topology are assumed. Perhaps less well known, but of great importance in the sequel is the Parrott theorem.

The Parrott Theorem . *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces with decompositions $\mathcal{H}_0 \oplus \mathcal{H}_1$, respectively $\mathcal{K}_0 \oplus \mathcal{K}_1$ and let M_X be the bounded transformation from $\mathcal{H}$ into $\mathcal{K}$ with operator matrix*

$$M_X = \begin{bmatrix} X & B \\ C & A \end{bmatrix} \tag{1}$$

with respect to the above decompositions. Then

$$\inf_X \|M_X\| = max\left\{ \left\| \begin{matrix} 0 & 0 \\ C & A \end{matrix} \right\|, \quad \left\| \begin{matrix} 0 & B \\ 0 & A \end{matrix} \right\| \right\} \tag{2}$$

This result first appeared in a paper of S. Parrott [25], in which it is used to obtain a generalization of the Nagy-Foias dilation theorem and interpolations theorems. As such, the expert will not be surprised at the utility this theorem has afforded us.

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Major Department: Mathematics

In this work, the representation of operators upon a Hilbert space endowed with a Nevanlinna-Pick reproducing kernel. A generalization of the commutant lifting theorem in this context. It is shown that in an appropriate category there are projective objects. Furthermore it is demonstrated that objects in this category have projective resolutions.

CHAPTER 1
INTRODUCTION

Our results concern Hilbert spaces endowed with a reproducing kernel $k(z,\zeta) = \sum_{n=0}^{\infty} a_n z^n \bar{\zeta}^n$ such that $a_0 = 1$, $a_n > 1$, $\frac{1}{k}(z,\zeta) = 1 - \sum_{n=1}^{\infty} b_n z^n \bar{\zeta}^n$, and $b_n \geq 0$ for $n \geq 1$. Following Agler [2] we call such a kernel k a Nevanlinna-Pick (NP) kernel.[1] A classical example of a Hilbert space endowed with an NP kernel is the Hardy space $\mathbb{H}_2$ with the Szegő kernel $k(z,\zeta) = \frac{1}{1-z\bar{\zeta}}$. In this case $a_n = 1$ for all $n \geq 0$, while $b_1 = 1$ and $b_n = 0$ for all $n \geq 2$. Classical results of complex analysis teach us that the Szegő kernel is a reproducing kernel for $\mathbb{H}_2$. The multiplication M_z defined via $f(z) \mapsto z \cdot f(z)$ defines an isometric operator on the space $\mathbb{H}_2$. It is also true that the set of analytic polynomials is dense in $\mathbb{H}_2$. In fact $\{z^n\}_{n=0}^{\infty}$ form an orthonormal basis for $\mathbb{H}_2$. Then relative to this basis, the effect of the multiplication operator M_z is to shift the Fourier coefficients "forward," hence the name shift operator. Now given a function $f \in \mathbb{H}_2$, we may define the map $M_f \mathbb{H}_2 \to \mathbb{H}_2$ via $g \mapsto fg$. If M_f is bounded, then we speak of the (multiplication) operator with symbol f. While it is clear that the operators M_f commute with the shift operator M_z, Beurling [5] showed that this essentially characterized the commutant of the shift operator M_z.

Another classical Hilbert space of interest is the Dirichlet space. Here our kernel is given by

$$k(z,\zeta) = \sum \frac{z^n \bar{\zeta}^n}{n+1}. \tag{1.1}$$

[1] Actually, we also require k to have a positive radius of convergence about $(0,0)$, and there exists a constant C such that $\frac{a_j}{a_{j+1}} \leq C^2$. The rationale for these conditions is found in Section 3.6.

The Dirichlet space is the completion of the pre-Hilbert space consisting of analytic polynomials endowed with the bilinear form

$$< z^s, z^t >= \delta_{st}(s+1). \tag{1.2}$$

Agler [2] has shown that the Dirichlet kernel defined in 1.1 is in fact an NP kernel for the Dirichlet space. Again, multiplication by z is a bounded operator on the Dirichlet space.

One is tempted then to establish *mutatis mutandis* much of the same body operatory theory for the Dirichlet kernel that exists for the Szegő kernel. Indeed, we believe this idea visible in much of what follows, although the following (counter) example cautions us to choose the generalizations of the Szegő kernel judiciously. Specifically, if we consider the Bergmann kernel where

$$k(z,\zeta) = \sum (n+1) z^n \bar{\zeta}^n, \tag{1.3}$$

then we see that k is not an NP kernel. In fact one can explicitly calculate that $b_1 = 2$ while $b_2 = -1$. In the sequel, we will see that the non-negativity of the b_n for $n \geq 1$ is crucial in establishing our results.

We arrive at our results by first fixing an NP kernel $k(z,\zeta)$. We define a pre-Hilbert space on the space of analytic polynomials via the bilinear form

$$< z^s, z^t >= \frac{\delta_{st}}{a_s}. \tag{1.4}$$

We denote by $H^2(k)$ the completion of this pre-Hilbert space. Upon the Hilbert space $H^2(k)$ we define the operator $S_k(f) = zf$ to be multiplication by the polynomial z. We see that S_k is bounded (see footnote 1). Given $\mathcal{H}$, a complex separable Hilbert space, we then define the operator ${}_{\mathcal{H}}\mathcal{S}_k = I \otimes S_k$ on the space $\mathcal{H} \otimes H^2(k)$. Of particular interest to us are those Hilbert spaces $\mathcal{K}$ and operators $T \in \mathcal{L}(\mathcal{K})$ for which there is an intertwining $W : \mathcal{H} \otimes H^2(k) \to \mathcal{K}$ of ${}_{\mathcal{H}}\mathcal{S}_k$ and T. By intertwining of operators, say T on $\mathcal{H}$ and V on $\mathcal{K}$, we mean a bounded linear map $W : \mathcal{H} \to \mathcal{V}$ such that $WT = VW$. One of our fundamental results is

Theorem . *Let $\mathcal{N}, \mathcal{H}$ and $\mathcal{M}$ be complex separable Hilbert spaces and $C \in \mathcal{L}(\mathcal{N})$. If there is a partial isometry $P : \mathcal{H} \otimes H^2(k) \to \mathcal{N}$ which intertwines the operators ${}_{\mathcal{H}}S_k$ and C, then for every bounded intertwining $f : \mathcal{M} \otimes H^2(k) \to \mathcal{N}$ there exists a bounded intertwining $F : \mathcal{M} \otimes H^2(k) \to \mathcal{H} \otimes H^2(k)$ of the operators ${}_{\mathcal{M}}S_k$ and ${}_{\mathcal{H}}S_k$ such that $\|F\| \leq \|f\|$ and $PF = f$.*

Diagrammatically this is represented as

$$\begin{tikzcd} & \mathcal{M} \otimes H^2(k) \arrow[dl, dotted, "F"'] \arrow[d, "f"] \\ \mathcal{H} \otimes H^2(k) \arrow[r, "P"] & N \end{tikzcd} \tag{1.5}$$

This leads directly to Corollary 3.8.4, and Corollary 3.8.5 which we see as a generalization of the commutant lifting theorem. We then establish

Theorem . *Let $\mathcal{M}$ be a Hilbert space endowed with an operator C from $\mathcal{L}(\mathcal{M})$. Suppose that there is a surjective partial isometry $P_{\mathcal{N}} : \mathcal{H}^2(k) \twoheadrightarrow \mathcal{M}$ which intertwines the operators C and ${}_{\mathcal{H}}S_k$. Let $\mathcal{K}$ denote the kernel of the partial isometry $P_{\mathcal{N}}$. Then there is a partial isometry $P_{\mathcal{K}} : H^2(k) \otimes \mathcal{K} \to \mathcal{K}$ which intertwines the operators ${}_{\mathcal{K}}S_k$ and ${}_{\mathcal{H}}S_k|_{\mathcal{K}}$.*

The homological importance of this last result is that we will be able to show that objects will have projective resolutions. In the classical case of the forward shift on the Hilbert space $\mathbb{H}_2$, this result reduces to the observation that upon restricting an isometry to an invariant subspace, the restricted operator is again an isometry. Thus when our kernel is the Szegő kernel, we obtain a proof that every object will have a two step projective resolution. In other words we obtain a proof that the $\mathbf{Ext}^n$ for $n \geq 2$ functors are trivial. Achieving the same level of knowledge when the Szegő kernel is replaced by the Dirichlet kernel has proven to be a challenging problem which at the present remains open.

In what follows we briefly describe the order of presentation. In Chapter 2 we introduce the necessary Homological Algebra we will require. The treatment is very specific to our needs and we only establish that part of the theory which we will later employ. In particular, in Section 2.1 we define a category, and products. No discussion is made of more general limits. We then define an additive category, in order to describe chain complexes and homotopy. A significant development in this material is the treatment of exact sequences, which we briefly explain. We show that one may decree a class of sequences to be exact. Once done, we can define a projective object, and establish what is meant by an acyclic chain complex. We then establish the solvability of two mapping problems which arise in Section 2.2. It is in Section 2.2 where we establish that projective resolutions (relative to our definiton of an exact sequence) are essentially unique. More precisely, projective resolutions are unique up to a homotopy equivalence. This uniqueness then allows the definiton of the **Ext** functor in Section 2.3.

Chapter 3 contains the operator theoretic results described above. Following some introductory remarks which place the results in context, we define a reproducing kernel Hilbert space in Section 3.5, and then define an NP kernel. Given an NP kernel k, we define the Hilbert space $H^2(k)$, which by its construction will be endowed with the given NP kernel as a reproducing kernel. In Section 3.6 we define our model operator ${}_{\mathcal{H}}\mathcal{S}_k$ and show that it is a bounded operator on the Hilbert space $\mathcal{H} \otimes H^2(k)$, where $\mathcal{H}$ is a complex separable Hilbert space. Section 3.7 contains technical results necessary for the proof of the theorems in Section 3.8.

In Sections 3.8 and 3.9 are found the statements and proofs of the results to which we give homological meaning in Chapter 4. We use the Parrott Theorem in the proof of Theorem 3.8.3, from which we are able to establish Corollary 3.8.4. We show in Chapter 4 that Corollary 3.8.4 implies the objects of the form $\mathcal{H} \otimes H^2(k)$ are projective in the category $\mathfrak{H}2$ defined in Section 4.2. Theorem 3.8.3 also allows

us to establish Corollary 3.8.5, which is our generalization of the commutant lifting theorem. In Section 3.9 we establish the existence of several limits of sequences of operators in the strong operator topology. The operators so defined are then used in the proof of Theorem 3.9.9.

In Chapter 4 we establish the category in which we work and then give the homological meaning of some of our results. Notably, we show that Theorem 3.9.9 demonstrates that projective resolutions exist for every object in the category $\mathfrak{H}2$ in which we work. These results show that an **Ext** functor can then be defined. Chapter 5 addresses specific questions that remain to be answered, as well as directions for continuation of the program established herein.

CHAPTER 2
HOMOLOGICAL ALGEBRA

In the sequel, some of the results of Chapter 3 will be given homological meaning. The homological algebra introduced here and used later, is standard with one significant exception. In the category in which we work, we will declare a certain class of sequences to be *exact*. It is relative to this notion of an exact sequence that subsequent homological results will be stated. The development of this material is provided for completeness. References for all of the material in this section are [20], [6], [30], and [36].

2.1 Foundations

We begin with the

Definition 2.1.1. A category $\mathfrak{C}$ consists of:

1. A class $ob\,\mathfrak{C}$ of objects.

2. For each ordered pair of objects (M, N), there is a set, written $hom(M, N)$. The elements of $hom(M, N)$ are called morphisms with domain M and codomain N. Furthermore, if $(M, N) \neq (O, P)$ then $hom(M, N)$ is disjoint from $hom(O, P)$.

3. For each ordered triple of objects M, N, O, there is a map, called composition, from $hom(M, N) \times hom(N, O) \rightarrow hom(M, P)$, which is associative.

4. Lastly, for every object M, there is a morphism $1_M \in hom(M, M)$ satisfying the following:

(a) For every object N and for every morphism $g \in hom(M, N)$, we have $1_M g = g$.

(b) For every object N and for every morphism $f \in hom(N, M)$' we have $f 1_M = f$.

We shall also require the

Definition 2.1.2. Let M_1, M_2 be objects from a category $\mathfrak{C}$. A product of M_1 and M_2 is an object M from $\mathfrak{C}$, along with morphisms p_1, and p_2 from $hom(M, M_1)$ and $hom(M, M_2)$, respectively, such that for every object N from $\mathfrak{C}$ and morphisms $f_i \in hom(N, M_i)$, there is a morphism $f \in hom(N, M)$ such that $p_i f = f_i$ for $i = 1, 2$.

Definition 2.1.3. By an additive category $\mathfrak{A}$, we mean a category $\mathfrak{A}$, such that

1. every finite set of objects has a product,
2. for every pair of objects (M, N), the set $hom(M, N)$ is endowed with a binary operation making $hom(M, N)$ an abelian group, and
3. the composition in 3, Definition 2.1.1 above, is $\mathbb{Z}$-bilinear.

Given an additive category $\mathfrak{A}$, a sequence of objects $M = (M_n)_{n \in \mathbb{Z}}$ is said to be *graded* or *a graded object.* A map ρ of degree r, between two graded objects M and N, is a sequence of morphisms $\rho = (\rho_n)$ such that $\rho_n : M_n \to N_{n+r}$. By a *chain complex* from $\mathfrak{A}$, we mean a graded object C together with a map of degree -1, $d : C \to C$, such that $d^2 = 0$. Here 0 stands for the identity element from each group $hom(C_n, C_{n-1})$.

Definition 2.1.4. Given two chain complexes (C, d) and (C', d'), a map $f : C \to C'$ of degree 0 is said to be a chain map if

$$d' f = f d. \tag{2.1}$$

Given two chain maps f and g between (C, d) and (C', d'), we say a map h of degree 1 is a chain homotopy if

$$d'h + hd = f - g. \tag{2.2}$$

In this case we say f and g are homotopic.

Given a chain map $f : C \to C'$, we say f is a homotopy equivalence if there is a chain map $f' : C' \to C$ such that ff' and $f'f$ are homotopic to the identity maps on C' and C, respectively. Let $\mathcal{E}$ be a class of sequences from the additive category $\mathfrak{A}$, each of the form

$$E' \xrightarrow{\epsilon'} E \xrightarrow{\epsilon} E''.$$

In particular each element of $\mathcal{E}$ determines a quintuple of three objects and two morphisms. Declare the elements of $\mathcal{E}$ to be *exact.* A chain complex (C, d) such that for every $n \in \mathbb{Z}$ the sequence

$$C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1}$$

is an element of $\mathcal{E}$ is said to be acyclic. If it is the case that the sequence $\{C_n\}$ is indexed by $\mathbb{N}$ or some finite index set, then by acyclic we understand that each consecutive triple of objects and the connecting morphisms are an element of $\mathcal{E}$. We now make the

Definition 2.1.5. Let P be an object of $\mathfrak{A}$. If for every sequence $E' \to E \to E''$ from $\mathcal{E}$ and morphism $\phi : P \to E$, such that the following diagram commutes

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, dotted, "\psi"'] \arrow[d, "\phi"] \arrow[dr, "0"] & \\ E' \arrow[r, "\epsilon'"] & E \arrow[r, "\epsilon"] & E'' \end{tikzcd} \tag{2.3}$$

there exists a morphism $\psi : P \to E'$ which preserves the commutivity of the diagram, then we say P is a projective object in $\mathfrak{A}$ relative to $\mathcal{E}$.

The following two results will establish the solution to mapping problems which arise in the next section.

Lemma 2.1.6. *Suppose that P is a projective object in $\mathfrak{A}$ relative to $\mathcal{E}$, a class of exact sequences. Given the diagram*

$$\begin{tikzcd} P \arrow[r, "d"] \arrow[d, dotted, "g"'] & Q \arrow[d, "f"] & \\ E' \arrow[r, "\epsilon'"] & E \arrow[r, "\epsilon"] & E'' \end{tikzcd} \tag{2.4}$$

with $\epsilon f d = 0$, and the bottom row exact, there exists a morphism $g : P \to E'$ which makes the diagram commute.

Proof. Apply definition 2.1.5, taking ϕ in equation (2.3) as fd. The result follows.

□

Lemma 2.1.7. *Suppose that P is a projective object in $\mathfrak{A}$ relative to $\mathcal{E}$, a class of exact sequences. Given the diagram, not necessarily commutative,*

$$\begin{tikzcd} & P \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dotted, "k"'] & Q \arrow[dl, "h"] \\ E' \arrow[r, "\epsilon'"] & E \arrow[r, "\epsilon"] & E'' \end{tikzcd} \tag{2.5}$$

where $\epsilon h d = \epsilon f$, and the bottom row is exact, then there is a morphism $k : P \to E'$ such that

$$\epsilon' k + hd = f \tag{2.6}$$

Proof. Apply definition 2.1.5, taking ϕ in equation (2.3) as $f - hd$. The result then follows. □

2.2 Resolutions

The material in Section 2.1 provides the necessary tools to establish a basic result in homological algebra, the fundamental theorem of homological algebra. The precise statement will be given below, but paraphrased, the theorem states that projective resolutions are unique. Just what unique means, as well as what a resolution is, will now be addressed. First we have

Definition 2.2.1. An object M in a category $\mathfrak{C}$ is said to be an initial object, if for every object $X \in \mathfrak{C}$ we have $hom(M, X)$ a singleton. On the other hand, if for every object $X \in \mathfrak{C}$ we have $hom(X, M)$ a singleton, then we say M is a terminal object. If M is both an initial and a terminal object, then we say M is a zero object. Zero objects are usually written as simply 0.

In the most common categories, zero objects are readily available. For example in the category of groups, the trivial group is a zero object. Likewise in the category of complex vector spaces, the trivial vector space is a zero object. The reason for the attention paid to zero objects is their appearance in

Definition 2.2.2. Let M be an object in an additive category $\mathfrak{A}$ with a class $\mathcal{E}$ of exact sequences. A resolution of M in $\mathfrak{A}$ relative to $\mathcal{E}$ is an acyclic sequence

$$\cdots \longrightarrow E_2 \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\epsilon} M \longrightarrow 0 \tag{2.7}$$

Arbitrary resolutions prove to be uninteresting. If we insist that there exist j such that for each element E_i with $i \geq j \geq 0$ *projective*, then we will find that such a resolution attains some measure of uniqueness. If E_i is projective for every $i \geq 0$, then the resolution will be called a projective resolution. The fundamental theorem of homological algebra follows.

Theorem 2.2.3. *Let (C, d) and (C, d') be chain complexes in an additive category $\mathfrak{A}$, endowed with a class $\mathcal{E}$ of exact sequences, and r be an integer. Suppose that*

$\{f_i : C_i \to C'_i\}_{i\leq r}$ is a sequence of morphisms such that $d'_i f_i = f_{i-1}d_i$ for $i \leq r$. If C_i is projective for $i > r$ and $C'_{i+1} \to C'_i \to C'_{i-1}$ is exact for $i \geq r$, then $\{f_i\}$ can be extended to a chain map $f : C \to C'$. Moreover this extension is unique in the sense that any other extension $\tilde{f}$ will be homotopic to f by a homotopy h such that $h_i = 0$ for $i \leq r$.

Proof. We proceed by induction. Let $n \geq r$ and assume that f_i has been defined for $i \leq n$ in such a way so that $d'_i f_i = f_{i-1}d_i$ for $i \leq n$. Consider the following mapping problem:

$$\begin{tikzcd} C_{n+1} \arrow[r, "d"] \arrow[d, dotted, "f_{n+1}"'] & C_n \arrow[r, "d"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_{n+1} \arrow[r, "d'"] & C'_n \arrow[r, "d'"] & C'_{n-1} \end{tikzcd} \tag{2.8}$$

The existence of f_{n+1} is given by Lemma 2.1.6.

The existence of chain map f extending $\{f_i\}_{i\leq r}$ now established, assume that g is another such chain map extending $\{f_i\}_{i\leq r}$. Again we proceed by induction to establish the homotopy between f and g. Let $n \geq r$ and assume that h_i has been defined for $i \leq n$ such that $d'h_i + h_{i-1}d = f_i - g_i$. In the event that $n = r$ take $h_i = 0$ for all $i \leq r$. Let $t_n = f_n - g_n$ for all n. Consider the following mapping problem

$$\begin{tikzcd} & C_{n+1} \arrow[r, "d"] \arrow[d, "t_{n+1}"] \arrow[dl, dotted, "h_{n+1}"] & C_n \arrow[r, "d"] \arrow[d, "t_n"] \arrow[dl, "h_n"] & C_{n-1} \arrow[dl, "h_{n-1}"] \\ C'_{n+2} \arrow[r, "d'"] & C'_{n+1} \arrow[r, "d'"] & C'_n & \end{tikzcd} \tag{2.9}$$

The existence of h_{n+1} is given by Lemma 2.1.7. □

Given two projective resolutions of an object M, we can apply Theorem 2.2.3 and conclude that the two resolutions are equivalent, in that there is a homotopy equivalence between the two projective resolutions. This then establishes the

uniqueness of a projective resolution. In the sequel we will consider constructions based upon a particular projective resolution. It can be shown that the constructions remain unchanged if the resolution is replaced by one to which it is homotopy equivalent. Thus, the constructions depend only upon the existence of a projective resolution and not the particular resolution used. This outline is justified by

Theorem 2.2.4. *Given projective resolutions P and P' of an object M from an additive category $\mathfrak{A}$ endowed with a class E of exact sequences, there is a chain map $f : P \to P'$, unique up to homotopy, which is a homotopy equivalence between P and P'.*

Proof. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dotted] & P_1 \arrow[r] \arrow[d, dotted] & P_0 \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & P'_2 \arrow[r] \arrow[d, dotted] & P'_1 \arrow[r] \arrow[d, dotted] & P'_0 \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd} \tag{2.10}$$

Two successive applications of Theorem 2.2.3 establish the chain maps between P and P', and P' and P necessary for a homotopy equivalence. □

2.3 The Ext Functor

This section contains the constructions alluded to in Section 2.2. Given an object M in an additive category $\mathfrak{A}$ endowed with a class of exact sequences $\mathcal{E}$, and a projective resolution P of M, we will define a functor from $\mathfrak{A}$ into the category of

abelian groups. Let N be an object from $\mathfrak{A}$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & N \arrow[r] & N \arrow[r] & N \arrow[r] & N \arrow[r] & 0
\end{tikzcd} \tag{2.11}$$

which induces the sequence

$$hom(P_0, N) \to hom(P_1, N) \to hom(P_2, N) \to \cdots \tag{2.12}$$

Since $\mathfrak{A}$ is an additive category, the hom sets are abelian groups, in particular the sequence (2.12) is a sequence in the abelian category of $\mathbb{Z}$ modules. The point being made here is that kernels, cokernels, and all finite limits in the category exist. So, the cohomology of the sequence (2.12) is well defined. The cohomology groups of the sequence (2.12) are called the Ext groups. The functor $Ext^n(M, -)$ from $\mathfrak{A}$ to the category of abelian groups assigns to each object N the n^{th} homology group of the sequence (2.12). The proof that this construction does not depend on the particular choice of projective resolution can be found in Jacobson [20].

CHAPTER 3
OPERATOR THEORY

3.1 Introductory Remarks

The seminal paper of D. Sarason [28] excited the interest of the operator theory community and at the same time provided the theoretical foundation for the eventual development of $\mathbb{H}^\infty$-control theory by G. Zames [37], J.W.Helton [18], A. Tannenbaum [35], C. Foias [15], and others in the 1980s. It was in fact work by B. Sz. Nagy and C. Foias [34, 33], and R.G. Douglas, P.S. Muhly, and C. Pearcy [10] in which the commutant lifting theorem was developed, providing geometrical insight into Sarason's results. The use of the commutant lifting theorem to solve interpolation problems relevant to control theory has been championed by C. Foias in [16, 14]. The success of the commutant lifting approach to the many and diverse problems to which it has been applied deserves emphasis. We will see that consideration of the commutant lifting theorem invites homological questions; subsequent chapters will address these questions.

Interpolation problems have a rich history of their own, independent of applications to control theory. R. Nevanlinna [23] and G. Pick [26], as well as C. Carathéodory, L. Fejér, and I. Schur, studied various problems of interpolating data with analytic functions. D. Sarason [28] is credited with providing the operator theoretic interpretation of these problems. In the sequel we will consider the Nevanlinna-Pick interpolation problem in the classical setting and its solution. Reproducing kernel Hilbert spaces will be defined, and the notion of a complete NP kernel [21] will be introduced. The introduction of a complete NP kernel will then

allow the definition of the function spaces that will be of primary interest in all that follows.

3.2 Classical Hardy Spaces

We introduce the following standard notation. Let $\mathbb{T}$ be the unit circle in the complex plane and m denote (normalized) Lebesgue measure. For $1 \leq p \leq \infty$ let $L^p(m)$ denote the classical function spaces on the unit circle. Let $\mathbb{H}^2$ denote the Hardy space of analytic functions on the open unit disk $\mathbb{D}$ which have square summable power series, and let $\mathbb{H}^\infty$ denote the space of functions in $\mathbb{H}^2$ which are bounded on $\mathbb{D}$. Both $\mathbb{H}^2$ and $\mathbb{H}^\infty$ can be identified with subspaces of $L^2(m)$ and $L^\infty(m)$, respectively, and we utilize these identifications as is convenient. The bilateral shift operator U is a unitary operator defined on L^2 via $Uf(z) = zf(z)$. $\mathbb{H}^2$ is invariant for U and the restriction of U to $\mathbb{H}^2$ is denoted by S. We refer to S as the *unilateral* shift operator, or simply the shift operator, and note that S is an isometry. These spaces have enjoyed the attention of a diverse audience, including both operator theorists and specialists in control theory. There is a wealth of material written about the Hardy spaces, and we refer the interested reader to the excellent works by P.L. Duren [12] and K. Hoffman [19]. The fact that $\mathbb{H}^2$ is invariant for U cannot be overstressed as can be seen in the following theorem first demonstrated by A. Beurling [5].

Beurling-Lax-Helson Theorem . *If $\mathcal{H}$ is a subspace of L^2 invariant with respect to the operator U, then there exist but two possibilities:*

1. *$U\mathcal{H} = \mathcal{H}$, in which case the is an $m-$measurable subset $A \subset \mathbb{T}$, such that $\mathcal{H} = \chi_A L^2$, where χ_A is the characteristic function of A.*

2. *$U\mathcal{H} \neq \mathcal{H}$, in which case there is a measurable function θ on $\mathbb{T}$ with $|\theta| = 1$ (a.e.), such that $\mathcal{H} = \theta\mathbb{H}^2$.*

For a proof we refer the reader to N.K. Nikol'skiĭ [24]. Our immediate interest in the Hardy Spaces stems from

Pick's Theoerem . *Given* $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{D}$*, and* $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{C}$*, there exists* $\phi \in \mathbb{H}^\infty$ *with* $\|\phi\|_\infty \leq 1$ *such that* $\phi(x_i) = z_i$ *for* $1 \leq i \leq n$ *if and only if the matrix*

$$\left[\frac{1 - z_i z_i^*}{1 - x_i x_i^*}\right] \tag{3.1}$$

is positive.

Sarason demonstrates that this theorem can be obtained as a special case of his Theorem 1 [28]:

Sarason's Theorem . *Let* ψ *be a nonconstant inner function,* S *as above, and* $\mathcal{K} = \mathbb{H}^2 \ominus \psi\mathbb{H}^2$*. If* T *is an operator that commutes with the projection of* S *onto* $\mathcal{K}$*, then there is a function* $\phi \in \mathbb{H}^\infty$ *such that* $\|\phi\| = \|T\|$ *and* $\phi(S) = T$ *where* $\phi(S)$ *denotes the projection onto* $\mathcal{K}$ *of the operator of multiplication on* L^2 *by* ϕ*.*

Indeed, let ψ be a finite Blaschke product with distinct zeros $\{x_1, x_2, \ldots, x_n\}$. Let $\mathcal{K}$ be the space $\mathbb{H}^2 \ominus \psi\mathbb{H}^2$. By the Beurling-Lax-Helson (BLH) theorem, we have that $\mathcal{K}$ is semi-invariant for the shift operator S. In fact we have an explicit description of $\mathcal{K}$ as the n-dimensional span of the functions $g_k(z) = \frac{1}{1 - x_k^* z}$ for $1 \leq k \leq n$. Saratson points out that an operator T on $\mathcal{K}$ commutes with the compression of S to $\mathcal{K}$ if and only if g_k is an eigenvector of T^* for $1 \leq k \leq n$. If we then define the operator T by $T^* g_k = z_k^* g_k$ for $1 \leq k \leq n$ (where z_k^* is the complex conjugate of z_k), then Sarason's theorem guarantees a $\phi \in \mathbb{H}^\infty$ with $\|\phi\|_\infty = \|T\|$, and such that the compression of multiplication by ϕ to $\mathcal{K}$ is identical to the action of T on $\mathcal{K}$. Since the functions g_k are in fact the kernel functions for evaluation at x_k, it is apparent then that $\phi(x_k) = z_k$ for $1 \leq k \leq n$. The requirement in Pick's theorem that the interpolating function ϕ have $\mathbb{H}^\infty$ norm less than or equal to 1 is

then equivalent to the operator T being a contraction. This latter condition is just the requirement that (3.1) is positive.

3.3 Dilations

Sarason's approach to interpolation focuses our attention on the space $\mathcal{K} = \mathbb{H}^2 \ominus \psi\mathbb{H}^2$ and the operator T defined by $T^* g_k = z_k^* g_k$ for $k = 1, \ldots, n$. As we saw in Section 3.2, the action of the adjoint T^* upon the vectors $\{g_k\}$ was fated by the requirement that T commute with the compression of the shift S to $\mathcal{K}$. In this section we consider more closely the relation between an operator and its projection or compression onto a semi-invariant subspace. More precisely, we illustrate circumstances under which an operator can be realized as just such a compression. Of this point of view, N.K. Nikol'skiĭ [24, page 2] writes

> The basic idea of the new non-classical spectral theory is to abstain from looking at the linear operator as a sum of simple transformations (of the type Jordan blocks) and instead consider it as "part" of a complicated universal mapping, allotted (in compensation) with many auxiliary structures.

The standard reference for the following material has become the work by B. Sz.-Nagy and C. Foias [32].

Definition 3.3.1. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces such that $\mathcal{H} \subset \mathcal{K}$. Given two operators $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{K} \to \mathcal{K}$, we say that B is a dilation of A if the following holds

$$AP = PB, \tag{3.2}$$

where P is the orthogonal projection of B onto A.

Let B, B' be two dilations of A acting on $\mathcal{K}, \mathcal{K}'$, respectively. If there is a unitary operator $\phi : \mathcal{K} \twoheadrightarrow \mathcal{K}'$ such that

1. $\phi(h) = h \ \forall h \in \mathcal{H}$, and

2. $\phi^{-1}B'\phi = B$,

then we say the two dilations $B : \mathcal{K} \to \mathcal{K}$, and $B' : \mathcal{K}' \to \mathcal{K}'$ are isomorphic. One of the best known results on dilations is

The Nagy Dilation Theorem . *Given C a contractive operator on a Hilbert space $\mathcal{H}$, there exists a Hilbert space $\tilde{\mathcal{H}}$ and an isometry $U : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ such that U is a dilation of C. Moreover this dilation U may be chosen to be minimal in the sense that*

$$\tilde{\mathcal{H}} = \bigvee_{0}^{\infty} U^n \mathcal{H}. \tag{3.3}$$

This minimal isometric dilation of C is then determined up to isomorphism.

In fact, P. Halmos [17] showed that a contraction can be dilated to a unitary operator. B. Sz.-Nagy [31] proved the following

The Nagy Dilation Theorem II . *Given C a contractive operator on a Hilbert space $\mathcal{H}$, there exists a Hilbert space $\tilde{\mathcal{H}}$ and a unitary operator $U : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ such that U is a dilation of C. Moreover this dilation U may be chosen to be minimal in the sense that*

$$\tilde{\mathcal{H}} = \bigvee_{-\infty}^{\infty} U^n \mathcal{H}. \tag{3.4}$$

This minimal unitary dilation of C is then determined up to isomorphism.

Proofs of both of these theorems are found in B. Sz.-Nagy and C. Foias [32]. The significance of the Nagy dilation theorem (NDT) for what follows is that

1. Contractions dilate to isometries, and

2. The restriction of an isometry to an invariant subspace is again an isometry.

It is precisely the homological perspective we assume which gives *categorical* significance to the above two points. This same perspective is assumed in the work of R. Douglas and V. Paulsen [11], S. Ferguson [13], as well as J.F. Carlson and D.N. Clark [8].

3.4 The Commutant Lifting Theorem

We begin by introducing the following notation. Let $\mathcal{K}_i$, for $i = 1, 2$, be Hilbert spaces, and let $T_i : \mathcal{K}_i \to \mathcal{K}_i$ be operators on these Hilbert spaces. By an intertwining of $\mathcal{K}_1$ and $\mathcal{K}_2$, we mean a bounded linear map $A : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$AT_1 = T_2A. \tag{3.5}$$

While we speak of an interwining of Hilbert spaces, equation (3.5) requires the map A to interact with the operators T_i, for $i = 1, 2$ in a specific fashion. The operators T_i for $i = 1, 2$, with which the intertwining A must interact via (3.5), will always be clear from context. Note that stated in the above language, the NDT tells us that given a contraction C on a Hilbert space $\mathcal{H}$, there exist an isometry U acting on a Hilbert space $\tilde{\mathcal{H}}$, such that $\mathcal{H} \subset \tilde{\mathcal{H}}$, and the orthogonal projection $P : \tilde{\mathcal{H}} \twoheadrightarrow \mathcal{H}$ intertwines C and UXS.

Shortly after Sarason's work [28] appeared, B. Sz.-Nagy and C. Foias [34], and then R.G. Douglas, P.S. Muhly, and C. Pearcy [10] offered what has come to be known as

The Commutant Lifting Theorem . *Given contractive operators C_1, C_2 acting on Hilbert spaces H_1, H_2, resp., and a bounded intertwining $A : H_1 \to H_2$, there exists a bounded intertwining $\tilde{A}$ of the minimal isometric dilations of C_1, C_2 such that $\|\tilde{A}\| \leq \|A\|$.*

Returning for the moment to the discussion of Pick's interpolations problem and Sarason's solution, we see that the commutant lifting theorem (CLT) can be used to provide a solution. Indeed, take $C_1 = C_2$ as the projection of the shift S onto the semi-invariant space $\mathbb{H}^2 \ominus \psi\mathbb{H}^2$. The matrix (3.1) is positive then if and only if there is a contractive intertwining of the compression of the shift S with itself. In the case where (3.1) is positive, the CLT provides a contractive intertwining of the

shift S with itself. The BLH theorem is then invoked to provide the existence of the function $\phi \in \mathbb{H}^\infty$ in the statement of Pick's theorem [16].

3.5 Reproducing Kernels

In section 3.2 we saw that the space $\mathcal{K} = \mathbb{H}^2 \ominus \psi\mathbb{H}^2$, where ψ was a finite Blashke product with distinct zeros $x_1, \ldots, x_n$ had a basis $g_1, \ldots, g_n$ where $g_k(z) = \frac{1}{1-x_k^* z}$ for $1 \leq k \leq n$. It was remarked then that these functions are precisely the kernel functions for evaluation at x_k. The importance of this fact, in particular to interpolation, will be brought to light in this section. For extensive coverage of material related to this section, the reader may consult N. Aronszajn [3], J. Burbea and P. Masani [7], S. Saitoh [27], J. Ball [4], J. Agler [2], and S. McCullough [22].

Let X be a set and $\mathcal{H}, \mathcal{B}$ be a Hilbert spaces. We denote the set of continuous linear maps from $\mathcal{H}$ into $\mathcal{B}$ by $\mathcal{L}(\mathcal{H}, \mathcal{B})$. In the case the domain and codomain coincide we write simply $\mathcal{L}(\mathcal{H})$. We make the

Definition 3.5.1. A Hilbert space $\mathcal{H}$ of functions $\{f | \ f : X \to \mathcal{B}\}$ is said to be a reproducing kernel Hilbert space if the the following hold:

1. $\{f_\beta | \beta \in \mathcal{B} \text{ and } f_\beta(x) = \beta \ \forall x \in X\} \subset \mathcal{H}$,
2. The map $\beta \mapsto f_\beta$ is bounded,
3. There exists a map $k : X \times X \to \mathcal{L}(\mathcal{B})$ satisfying the following:
   (a) For each $s \in X$, the map $k(\cdot, s) : \mathcal{B} \to \mathcal{H}$ via $\beta \mapsto k(\cdot, s)\beta$ is a bounded map, and
   (b) If $f \in \mathcal{H}$, $\beta \in \mathcal{B}$, and $s \in X$, then

$$< f, k(\cdot, s)\beta >=< f(s), \beta > . \tag{3.6}$$

If $\mathcal{H}$ is a reproducing kernel Hilbert space, the map k above will be referred to as the reproducing kernel, or just kernel if clear from context.

Theorem 3.5.2. *A Hilbert space $\mathcal{H}$ of functions $\{f \mid f : X \to \mathcal{B}\}$ satisfying properties (1) and (2) of Definition 3.5.1 is a reproducing kernel Hilbert space if and only if for each $s \in X$, the evaluation $(\beta, f) \mapsto < f(s), \beta >$ is a bounded linear functional on $\mathcal{B} \oplus \mathcal{H}$.*

Proof. Let $\mathcal{H}$ be a reproducing kernel Hilbert space. Then

$$| < f(s), \beta > | = | < f, k(\cdot, s)\beta > | \leq \|f\| \, \|k(\cdot, s)\beta\| \leq \|f\| \, \|k(\cdot, s)\| \, \|\beta\|, \tag{3.7}$$

since $k(\cdot, s) : \mathcal{B} \to \mathcal{H}$ is a bounded map, which demonstrates continuity at $0 \oplus 0 \in \mathcal{B} \oplus \mathcal{H}$. Linearity then guarantees that the evaluation is bounded.

Conversely, suppose that for each $s \in X$ the evaluation $(\beta, f) \mapsto < f(s), \beta >$ is a continuous map from $\mathcal{B} \oplus \mathcal{H}$ into $\mathbb{C}$. Fixing s and β we see that the Riesz representation theorem then guarantees that there is an element $k(\cdot, s)\beta \in \mathcal{H}$ such that

$$< f(s), \beta > = < f, k(\cdot, s)\beta > . \tag{3.8}$$

Continuity of the evaluation $< f(s), \beta >$ implies that there is a constant C_s such that $\|k(\cdot, s)\beta\| \leq C_s \|\beta\| \cdot \|f\|$. In particular for $t \in X$ we have

$$\|k(\cdot, t)\beta\|^2 \leq C_t \|k(\cdot, t)\beta\| \cdot \|\beta\| \tag{3.9}$$

hence

$$\|k(\cdot, t)\beta\| \leq C_t \|\beta\|. \tag{3.10}$$

Define $\hat{k} : X \times X \to \mathcal{B}^{\mathcal{B}}$ via $\hat{k}(s, t)\beta = k(s, t)\beta$. We claim that $\hat{k} : X \times X \to \mathcal{L}(\mathcal{B})$. Given β_1 and β_2 from $\mathcal{B}$, and w_1 and w_2 from $\mathbb{C}$, let $\gamma = w_1\beta_1 + w_2\beta_2$; then we have

$$\begin{aligned} < f, k(\cdot, s)w_1\beta_1 + k(\cdot, s)w_2\beta_2 > &= < f(s), w_1\beta_1 > + < f(s), w_2\beta_2 > \\ &= < f(s), \gamma > \\ &= < f, k(\cdot, s)\gamma > . \end{aligned} \tag{3.11}$$

Since this is true for all $f \in \mathcal{H}$, we have $k(\cdot, s)w_1\beta_1 + k(\cdot, s)w_2\beta_2 = k(\cdot, s)\gamma$, and hence in particular if we evaluate at $t \in X$, then we see that $\hat{k}(t, s)$ is indeed linear. Moreover, $\hat{k}(s, t)$ is bounded as

$$\begin{aligned}
\|\hat{k}(s,t)\beta\|^2 &= | < k(s,t)\beta, k(s,t)\beta > | \\
&= | < k(\cdot,t)\beta, k(\cdot,s)k(s,t)\beta > |
\end{aligned}$$

using (3.8). Continuity of the evaluation guarantees

$$\begin{aligned}
| < k(\cdot,t)\beta, k(\cdot,s)k(s,t)\beta > | leC_s\|k(\cdot,t)\beta\| \cdot \|k(s,t)\beta\| \\
= C_t\|\beta\|C_s\|k(s,t)\beta\|. \qquad (3.12)
\end{aligned}$$

Hence

$$\|k(s,t)\beta\| \leq C_tC_s\|\beta\| \qquad (3.13)$$

which shows that $\|\hat{k}(s,t)\| \leq C_sC_t$, and thus $\hat{k}(s,t) \in \mathcal{L}(\mathcal{B})$. □

Example 3.5.3. Let $\mathcal{H} = \mathbb{H}^2$, $X = \mathbb{D}$, and k be the Szegő kernel, $k(\eta, \xi) = \frac{1}{1-\eta\bar{\xi}}$. Then $\mathbb{H}^2$ is a reproducing kernel Hilbert space when endowed with the standard inner product.

In fact, J. Agler [2] and S. McCullough [22] have shown that the existence of a reproducing kernel allows one to recover (operator valued) versions of Nevanlinna-Pick interpolation problems. For the moment our interest lies in considering the the map k alone. In our approach we assume our kernel has the form

$$k(z, \zeta) = \sum_0^\infty a_n z^n \overline{\zeta}^n, \qquad (3.14)$$

where $a_0 = 1$, and $a_n > 0$. We also assume k has a positive radius of convergence about (0,0) and

$$\frac{a_j}{a_{j+1}} \leq C^2. \qquad (3.15)$$

Since $k(0,0) = 1$, near $(0,0)$ we have

$$\frac{1}{k}(z,\zeta) = 1 - \sum_{n=1}^{\infty} b_n z^n \overline{\zeta}^n, \tag{3.16}$$

and we note for future use that, for $n \geq 1$,

$$a_n = \sum_{s=1}^{n} b_s a_{n-s}. \tag{3.17}$$

In this context we make the following

Definition 3.5.4. We say k is an NP kernel if $b_n \geq 0$ for all $n \geq 1$.

Example 3.5.5.

1. Let $a_n = 1$ for all n. Then k is the Szegő kernel described above. In this case $b_1 = 1$ and $b_n = 0$ for all $n \geq 2$, hence the Szegő kernel is an NP kernel.

2. Let $a_n = \frac{1}{n+1}$. Then k is the Dirichlet kernel. While true [1, 29], it is nontrivial to show that in this case k is an NP kernel.

3. Let $a_n = n + 1$. Then k is the Bergman kernel. In this case we see that $k(z,\zeta) = \frac{1}{(1-z\overline{\zeta})^2}$. One can then observe in this case that $b_1 = 2$, while $b_2 = -1$. Hence the Bergman kernel is *not* an NP kernel.

Given an NP kernel k we define a bilinear form on the set of analytic polynomials by

$$< z^s, z^t >= \begin{cases} \frac{1}{a_s}, & \text{if } s = t; \\ 0, & \text{if } s \neq t. \end{cases} \tag{3.18}$$

With deference then to example 3.5.5 we write $H^2(k)$ to denote the Hilbert space obtained as the completion of the pre-Hilbert space structure induced by equation (3.18). We will denote by $H^\infty(k)$ those $f \in H^2(k)$ which give rise to a bounded multiplication operator $M_f : H^2(k) \to H^2(k)$ with symbol f. In the sequel we will see that condition (3.15) implies that we can define an operator S_k on $H^2(k)$ via $f \mapsto zf$.

3.6 Tensor Products and The Model Operator

Let k be an NP kernel $k(z,\zeta) = \sum_0^\infty a_n z^n \overline{\zeta}^n$ and C be as in equation (3.15). For each $l \in \mathbb{N}$ define $s_l \in H^2(k)$ by $s_l = a_l z^l$.

Lemma 3.6.1. *Relative to the inner product* (3.18) *with which $H^2(k)$ is endowed, $\{s_l\}$ is a dual basis to $\{z^l\}$.*

Proof. It is clear from inspection that $\{s_l\}$ is an orthogonal set. Let $\mathcal{M}$ denote the linear manifold spanned by $\{s_l\}$. Let $h^* \in H^2(k)^*$ such that $h^*(s_l) = a_l h^*(z^l) = 0$ for all $l \in \mathbb{N}$. Since $H^2(k)$ is defined as the completion of the pre-Hilbert space induced by equation (3.18), the polynomials are dense in $H^2(k)$, hence $h^* = 0$, and therefore $\mathcal{M} = H^2(k)$. □

We define the operator $S_k : H^2(k) \to H^2(k)$ via $f \mapsto zf$ and note that S_k is bounded. Indeed, let $f = \sum_{n=0}^\infty c_n z^n \in H^2(k)$ and consider

$$\begin{aligned}
\|S_k f\|^2 &= \langle S_k f, S_k f\rangle \\
&= \langle \sum_{n=0}^\infty c_n z^{n+1}, \sum_{n=0}^\infty c_n z^{n+1} \rangle \\
&= \sum_{n=0}^\infty \frac{|c_n|^2}{a_{n+1}} \\
&= \sum_{n=0}^\infty \frac{|c_n|^2}{a_n} \frac{a_n}{a_{n+1}}.
\end{aligned} \tag{3.19}$$

Equation (3.15) then implies that $\|S_k f\| \le C\|f\|$. If for $l < 0$ we intrepret $s_l = 0$, then

$$\begin{aligned}
< z^{l-1}, S_k^* s_l > &=< S_k z^{l-1}, s_l > \\
&=< z^l, s_l >= 1,
\end{aligned} \tag{3.20}$$

hence

$$S_k^* s_l = s_l. \tag{3.21}$$

Let $\mathcal{M}$ be a Hilbert space. We denote $\mathcal{M} \otimes H^2(k)$ by $\mathcal{M}^2(k)$. Note that each element $f \in \mathcal{M}^2(k)$ can be written as

$$f = \sum m_n \otimes z^n, \tag{3.22}$$

for $m_n \in \mathcal{M}$, where the series converges in norm. Given operators $T : \mathcal{M} \to \mathcal{M}$ and $V : H^2(k) \to H^2(k)$, we write $T \otimes V$ for the operator on $\mathcal{M}^2(k)$ defined via $\sum m_n \otimes z^n \mapsto \sum Tm_n \otimes Vz^n$. In particular we denote by ${}_{\mathcal{M}}\mathcal{S}_k$ the operator $I \otimes S_k$ on $\mathcal{M}^2(k)$ via

$${}_{\mathcal{M}}\mathcal{S}_k f = \sum m_n \otimes z^{n+1}, \tag{3.23}$$

or if use is clear from context, we write $\mathcal{S}_k$ for ${}_{\mathcal{M}}\mathcal{S}_k$.

3.7 Fundamental Inequalities

In the exposition which follows we will often have the need to express the matrix of a linear transformation relative to a given basis in block form. We associate to a linear transformation a matrix relative to the closure of the linear manifold spanned by an orthogonal, but not necessarily normal, set of vectors. The adjoint of the transformation then has associated a matrix. This association is given by

Lemma 3.7.1. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $\{v_k\} \subset \mathcal{H}$, $\{w_k\} \subset \mathcal{K}$ be dense sets of mutually orthogonal vectors in $\mathcal{H}$ and $\mathcal{K}$, respectively. Let $T : \mathcal{H} \to \mathcal{K}$ be a linear transformation, and relative to the sets $\{v_k\}, \{w_k\}$ we associate to T the matrix*

$$(t_{ij}) = \left(\frac{< Tv_j, w_i >}{< w_i, w_i >}\right). \tag{3.24}$$

Lemma 3.7.2. *With the same notation as in Lemma 3.7.1, the matrix of the transpose map $T^* : \mathcal{K} \to \mathcal{H}$ is given by*

$$(\hat{t}_{ij}) = \left(\bar{t}_{ji}\frac{< v_j, v_j >}{< v_i, v_i >}\right). \tag{3.25}$$

Proof. Calculating we have

$$
\begin{aligned}
(\hat{t}_{ij}) &= \left(\frac{< T^*w_j, v_i >}{< v_i, v_i >}\right) \\
&= \left(\frac{< w_j, Tv_i >}{< v_i, v_i >}\right) \\
&= \left(\frac{\overline{< Tv_i, w_j >}}{< v_j, v_j >}\frac{< v_j, v_j >}{< v_i, v_i >}\right) \\
&= \left(\bar{t}_{ji}\frac{< v_j, v_j >}{< v_i, v_i >}\right) \qquad (3.26)
\end{aligned}
$$

□

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, $\{v_{ik}\} \subset \mathcal{H}_i$ for $i = 1, 2$, and $T : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{K}$. Define $T_j : \mathcal{H}_1 \to \mathcal{K}$ via $T_j(h_1) = T(h_1 \otimes v_{2j})$. In this case matrix (3.24) of Lemma 3.7.1 will be written as

$$(\dots\ T_{j-1}\ T_j\ T_{j+1}\ \dots) \qquad (3.27)$$

relative to the orthogonal decomposition

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \oplus_j \left(\mathcal{H}_1 \otimes [v_{2j}]\right). \qquad (3.28)$$

The matrix of the adjoint $T^* : \mathcal{K} \to \mathcal{H}_1 \otimes \mathcal{H}_2$ in accordance with Lemma 3.7.2 written as

$$\begin{pmatrix} \vdots \\ \frac{T_j^*}{\|v_{2j}\|^2} \\ \vdots \end{pmatrix}. \qquad (3.29)$$

Now given two complex sequences $c, d : \mathbb{N} \to \mathbb{C}$ we form the convolution $(c * d)_n = \sum_{j+k=n} c_j d_k$. The set of all sequences then forms a semigroup with identity $e_0 = 1$, $e_n = 0$ for all $n \geq 1$. Two sequences c, d such that $c * d = e$ are said to be an inverse pair.

For $\mathcal{K}_1 \subset \mathcal{K}_2$ two Hilbert spaces, let $T \in \mathcal{L}(\mathcal{K}_2)$ and $C \in \mathcal{L}(\mathcal{K}_1)$. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \to \mathcal{K}_1$, such that $TC^jA = T^{j+1}A$ for $j \geq 0$. Then we have the

Lemma 3.7.3. *With notation as above, let c and d be an inverse pair of sequences such that $d_j \geq 0$ for $j \geq 0$. If for all $N, M \in \mathbb{N}$ we have*

$$\sum_{j=0}^{N} c_j (C^j A)(C^j A)^* \leq I \tag{3.30}$$

and

$$\sum_{k=1}^{M} d_k T^k T^{*k} \leq I \tag{3.31}$$

then for all $M \in \mathbb{N}$,

$$I + \sum_{n=1}^{M} c_n d_0 (T^n A)(T^n A)^* \geq 0. \tag{3.32}$$

Proof. Computing we find

$$\begin{aligned} I &\geq \sum_{k=1}^{M} d_k T^k T^{*k} \\ &\geq \sum_{k=1}^{M} d_k T^k \left(\sum_{j=0}^{M-k} c_j (C^j A)(C^j A)^* \right) T^{*k} \\ &= \sum_{k=1}^{M} \sum_{j=0}^{M-k} d_k c_j T^{k+j} A A^* T^{*k+j}, \end{aligned} \tag{3.33}$$

since $T^{j+1} A = T C^j A$. Reindexing we have

$$I \geq \sum_{n=1}^{M} \sum_{l=1}^{n} d_l c_{n-l} T^n A A^* T^{*n} \tag{3.34}$$

which in view of the identity $c_n d_0 = \sum_{j=0}^{n-1} c_j d_{n-j}$ for $n \geq 1$ yields

$$I + \sum_{n=1}^{M} c_n d_0 (T^n A)(T^n A)^* \geq 0. \tag{3.35}$$

□

Let $\mathcal{M}$ be a Hilbert space, $\mathcal{M}^2(k) = \mathcal{M} \otimes H^2(k)$ be as in Section 3.6, $\mathcal{K}_1, \mathcal{K}_2$ as above, $f \in \mathcal{L}(\mathcal{M}^2(k), \mathcal{K}_1)$ such that

$$f_j = c_j C^j f_0. \tag{3.36}$$

where f_j is as in equation (3.27). Then $\|f\| \leq 1$ if and only if

$$I \geq ff^* = \sum a_j f_j f_j^* = \sum a_j |c_j|^2 C^j f_0 f_0^* C^{j*} \tag{3.37}$$

Suppose that there is a non-negative sequence $d : \mathbb{N} \to \mathbb{C}$ such that $d * ac = e$, where $ac_j = a_j |c_j|^2$. Under these conditions we have

Corollary 3.7.4. *If $T \in \mathcal{L}(\mathcal{K}_2)$ is a dilation of $C \in \mathcal{L}(\mathcal{K}_1)$ such that $T^2 P_{\mathcal{K}_1} = TC$, and*

$$I \geq \sum_{k=1}^{M} d_k T^k T^{*k} \tag{3.38}$$

for all $M \geq 1$, then there is a map $F \in \mathcal{L}(\mathcal{M}^2(k), \mathcal{K}_2)$ such that $P_{\mathcal{K}_1} F = f$, and $\|F\| \leq$ max $\{\|f\|, |d_0^{-1}| \, \|f\|\}$.

Proof. Without loss of generality assume that $\|f\| \leq 1$, hence equation (3.37) holds. Consider then the map $F : \mathcal{M}^2(k) \to \mathcal{K}_2$ whose matrix relative to the decomposition of $\mathcal{K}_2 = (\mathcal{K}_2 \ominus \mathcal{K}_1) \oplus \mathcal{K}_1$ has the form

$$F_0 = c_0 \begin{pmatrix} g_0 \\ f_0 \end{pmatrix} \quad F_j = c_j \begin{pmatrix} g_j \\ f_j \end{pmatrix} = c_j T^j f_0 \quad j \geq 1. \tag{3.39}$$

for some $g_0 : \mathcal{M} \to \mathcal{K}_2 \ominus \mathcal{K}_1$. Since T is a dilation of C we have $P_{\mathcal{K}_1} T^j = C^j P_{\mathcal{K}_1}$ for all $j \geq 0$, hence $P_{\mathcal{K}_1} F = f$. Moreover $\|F\| \leq 1$ if and only if

$$I \geq FF^* = \sum_{j=0} a_j F_j F_j^* \tag{3.40}$$

$$= a_0 |c_0|^2 F_0 F_0^* + \sum_{j=1} a_j |c_j|^2 T^j f_0 f_0^* T^{j*}. \tag{3.41}$$

Since $T^2 P_{\mathcal{K}_1} = TC$, we have by induction

$$T^j f_0 = TC^{j-1} f_0. \tag{3.42}$$

Recall that equation (3.37) holds, so we can apply Lemma 3.7.3 with $A = f_0$ to conclude that

$$\| \sum_{j=1} a_j |c_j|^2 d_0 T^j f_0 f_0^* T^{j*} \| \leq 1 \tag{3.43}$$

Thus the operator matrix

$$\begin{pmatrix} c_0 f_0 & c_1 f_1 & \cdots \end{pmatrix} \tag{3.44}$$

is bounded with norm $\|f\|$ and the operator matrix

$$\begin{pmatrix} c_1 T f_0 & c_2 T^2 f_0 & \cdots \end{pmatrix} \tag{3.45}$$

has norm bounded above by $|d_0^{-1}|$. The Parrott theorem can then be applied to the operator matrix

$$\begin{pmatrix} c_0 g_0 & c_1 g_1 & \cdots \\ c_0 f_0 & c_1 f_1 & \cdots \end{pmatrix} \tag{3.46}$$

to conclude that there exist g_0 such that

$$\|F\| \leq \max \{1, |d_0^{-1}|\}. \tag{3.47}$$

$\square$

3.8 Constructions

Let $\{\mathcal{M}_i\}_{i=1}^{\infty}$ be a sequence of Hilbert spaces such that $\mathcal{M}_{i+1} \supset \mathcal{M}_i$. Denote by p_i the orthogonal projection from $\mathcal{M}_{i+1}$ onto $\mathcal{M}_i$. Let $\mathcal{M}_{-1} = 0$, and let $\underset{\leftarrow}{\mathcal{M}}$ denote the Hilbert space $\oplus_{i=0}^{\infty}(\mathcal{M}_i \ominus \mathcal{M}_{i-1})$ whose elements are the vectors

$$\mathbf{m} = (m_0, m_1, \dots) \text{ with } \|\mathbf{m}\| = \sum_0^{\infty} \|m_i\|^2 \leq \infty. \tag{3.48}$$

We denote by P_i the projection from the Hilbert space $\underset{\leftarrow}{\mathcal{M}}$ to the space $\mathcal{M}_i$.

Lemma 3.8.1. *Let $\mathcal{H}$ be a Hilbert space and $f_i : \mathcal{H} \to \mathcal{M}_i$ be a sequence of bounded maps. If there is a constant C such that for all $i = 0, 1, 2, \dots$ we have $\|f_i\| \leq C$ and $p_i f_{i+1} = f_i$, then there is a bounded map $F : \mathcal{H} \to \underset{\leftarrow}{\mathcal{M}}$ such that $\|F\| \leq C$ and $P_i F = f_i$*

Proof. Define F via

$$h \mapsto \{f_0 h, (f_1 - f_0)h, \dots, (f_i - f_{i-1})h, \dots\}. \tag{3.49}$$

Since f_{i+1} is a dilation of f_i, $f_{i+1} - f_i$ maps h into $\mathcal{M}_{i+1} \ominus \mathcal{M}_i$. Inspection shows that

$$\|\{f_0 h, (f_1 - f_0)h, \dots, (f_i - f_{i-1})h, 0, \dots\}\| = \|f_i h\| \leq C\|h\| \text{ for all } i \geq 1. \tag{3.50}$$

hence $\|Fh\| \leq C\|h\|$. F is therefore well defined and bounded with $\|F\| \leq C$. □

Combining Corollary 3.7.4, and Lemma 3.8.1 we have

Theorem 3.8.2. *Let $\{\mathcal{M}_i\}_{i=1}^{\infty}$ be a sequence of Hilbert Spaces such that $\mathcal{M}_{i+1} \supset \mathcal{M}_i$, $T_i \in \mathcal{L}(\mathcal{M}_i)$, T_{i+1} is a dilation of T_i, and $T_{i+1}^2 P_{\mathcal{M}_i} = T_{i+1}T_i$. Let $\mathcal{H}$ be a Hilbert space, and $f : \mathcal{L}(\mathcal{H}^2(k), \mathcal{M}_0)$, such that*

$$f_j = c_j T_0^j f_0. \tag{3.51}$$

If there is a non-negative sequence $d : \mathbb{N} \to \mathbb{C}$ such that $d * ac = e$, where $ac_j = a_j c_j$, and equation (3.38) holds, then there is a map $F \in \mathcal{L}(\mathcal{H}^2(k), \underset{\leftarrow}{\mathcal{M}})$ with $\|F\| \leq \|f\|$ and $P_{\mathcal{M}_0} F = f$.

Let $\mathcal{H}$ and $\mathcal{M}$ be two Hilbert spaces, and $\mathcal{N}$ be a Hilbert space endowed with an operator $C \in \mathcal{L}(\mathcal{N})$.

Theorem 3.8.3. *With the above notation, suppose that there is a partial isometry $P : \mathcal{H}^2(k) \to \mathcal{N}$ which interwines the operators ${}_{\mathcal{H}}S_k$ and C. Then for every bounded intertwining $f : \mathcal{M}^2(k) \to \mathcal{N}$ there exists a bounded intertwining $F : \mathcal{M}^2(k) \to \mathcal{H}^2(k)$ such that $\|F\| \leq \|f\|$ and $PF = f$.*

Proof. If $P = 0$ then set $F = 0$. Hence assume $P \neq 0$. Let $f : \mathcal{M}^2(k) \to \mathcal{N}$ be an intertwining of ${}_{\mathcal{M}}S_k$ and C. Without loss of generality assume that $\|f\| \leq 1$. We proceed recursively. Define $\mathcal{N}_0 = P(\mathcal{N})$. Since P is a partial isometry, $P = PP^*P$ and thus P^* when restricted to the image of P is an isometry. Hence we abuse notation by referring to the subspace $P^*(\mathcal{N}_0) \subset \mathcal{H}^2(k)$ as $\mathcal{N}_0$. For $i \geq 1$ define j_i to be the least such integer such that $\mathcal{H} \otimes [k_{j_i}]$ is not a subspace of $\mathcal{N}_{i-1}$. Then we set $\mathcal{N}_i$

to be the closure of the span of the subspaces $\mathcal{N}_{i-1}$ and $\mathcal{H}\otimes[k_{j_i}]$. Let $C_0 = C|_{\mathcal{N}_0}$ and note that P intertwines ${}_{\mathcal{H}}S_k$ and C. Let $i \geq 1$, and for each $\mathcal{N}_i$ define an operator $C_i = P_{\mathcal{N}_i}\,{}_{\mathcal{H}}S_k$. Since ${}_{\mathcal{H}}S_k^*(\mathcal{H}\otimes[k_q]) \subset \mathcal{H}\otimes[k_{q-1}]$ we have $C_i(\mathcal{N}_i \ominus \mathcal{N}_{i-1}) = 0$. Hence $C_{i+1}^2 P_{\mathcal{N}_i} = C_{i+1}C_i$. One then verifies that $\underset{\leftarrow}{\mathcal{N}} \cong \mathcal{H}^2(k)$, and applies Theorem 3.8.2. □

Let $\mathcal{H}$ and $\mathcal{M}$ be as above, and $\mathcal{N}'$, $\mathcal{N}$, $\mathcal{N}''$ be Hilbert spaces endowed with operators C', C, C'' from $\mathcal{L}(\mathcal{N}')$, $\mathcal{L}(\mathcal{N})$, $\mathcal{L}(\mathcal{N}'')$, respectively. Assume further that there exist partial isometries $\pi' : \mathcal{N}' \to \mathcal{N}$ and $\pi : \mathcal{N} \to \mathcal{N}''$ which intertwine the operators C', C, and C'', and $\pi\pi' = 0$. Lastly suppose that there is a surjective partial isometry $P_{\mathcal{N}'} : \mathcal{H}^2(k) \twoheadrightarrow \mathcal{N}'$ which intertwines the operators C' and ${}_{\mathcal{H}}S_k$. Under these conditions we have

Corollary 3.8.4. *If $f \in \mathcal{L}(\mathcal{M}^2(k), \mathcal{N})$ interwines ${}_{\mathcal{M}}S_k$ and C and $\pi f = 0$, then there is an intertwining $F : \mathcal{M}^2(k) \to \mathcal{N}'$ such that $\|F\| \leq \|f\|$ and $\pi' F = f$.*

Proof. Apply Theorem 3.8.3 to the composition $\pi' P_{\mathcal{N}'}$ and the map

$$f \in \mathcal{L}(\mathcal{M}^2(k), \mathcal{N}).$$

Since $P_{\mathcal{N}'}$ is surjective the composition $\pi' P_{\mathcal{N}'}$ is a partial isometry. Indeed, let $x \in \ker(\pi' P_{\mathcal{N}'})^{\perp}$. In particular $x \in \ker(P_{\mathcal{N}'})^{\perp}$, since

$$\ker(P_{\mathcal{N}'}) \subset \ker(\pi' P_{\mathcal{N}'}).$$

Therefore we have

$$< P_{\mathcal{N}'}(x), P_{\mathcal{N}'}(x) > = < x, x > . \tag{3.52}$$

Moreover we have $P_{\mathcal{N}'}(\ker(\pi' P_{\mathcal{N}'})^{\perp}) \subset \ker(\pi')^{\perp}$. Indeed, $P_{\mathcal{N}'}$ is a surjective map, so for $y \in \ker(\pi')$ we compute

$$\begin{aligned} < P_{\mathcal{N}'}(x), y > &= < P_{\mathcal{N}'}(x), P_{\mathcal{N}'}(\tilde{y}) > \qquad P_{\mathcal{N}'}(\tilde{y}) = y \\ &= < x, \tilde{y} > \qquad \tilde{y} \in \ker(\pi' P_{\mathcal{N}'}) \\ &= 0. \end{aligned} \tag{3.53}$$

Hence

$$< \pi' P_{\mathcal{N}'}(x), \pi' P_{\mathcal{N}'}(x) > =< P_{\mathcal{N}'}(x), P_{\mathcal{N}'}(x) >$$
$$=< x, x >$$

establishing the claim. □

Let $\mathcal{H}$ and $\mathcal{M}$ be as above, and $\mathcal{N}'$, $\mathcal{N}$ be a Hilbert spaces endowed with operators C', C from $\mathcal{L}(\mathcal{N}')$, $\mathcal{L}(\mathcal{N})$, respectively. Lastly suppose that there are surjective partial isometries $P_{\mathcal{N}'} : \mathcal{H}^2(k) \twoheadrightarrow \mathcal{N}'$ and $P_{\mathcal{N}} : \mathcal{M}^2(k) \twoheadrightarrow \mathcal{N}$ which intertwine the operators C' and ${}_{\mathcal{H}}S_k$, and C and ${}_{\mathcal{M}}S_k$, respectively. Under these conditions we have

Corollary 3.8.5. *If there exist a map $g : \mathcal{N}' \to \mathcal{N}$ which intertwines the operators C' and C, then there is an intertwining $F : \mathcal{H}^2(k) \to \mathcal{M}^2(k)$ such that $\|F\| \leq \|g\|$ and $F^*|_{\mathcal{N}} = g^*$.*

Proof. Apply Theorem 3.8.3 to the composition $gP_{\mathcal{N}'} : \mathcal{H}^2(k) \to \mathcal{N}'$. Since $P_{\mathcal{N}} : \mathcal{M}^2(k) \twoheadrightarrow \mathcal{N}$ is a surjective partial isometry, Theorem 3.8.3 insures the existence of an intertwining $F : \mathcal{H}^2(k) \to \mathcal{M}^2(k)$, satisfying $\|F\| \leq \|gP_{\mathcal{N}'}\| \leq \|g\|$. Using the fact that $P_{\mathcal{N}}F = gP_{\mathcal{N}'}$, and taking adjoints then shows that $F^*|_{\mathcal{N}} = g^*$. □

3.9 Kernels

Let $\mathcal{H}$ be as above, and $\mathcal{N}$ be a Hilbert space endowed with an operator C from $\mathcal{L}(\mathcal{N})$. Suppose that there is a surjective partial isometry $P_{\mathcal{N}} : \mathcal{H}^2(k) \twoheadrightarrow \mathcal{N}$ which intertwines the operators C and ${}_{\mathcal{H}}S_k$. Under these conditons we make the

Definition 3.9.1. Let $D_n^C = I - \sum_{l=1}^n b_l C^l C^{*l}$. If in the event that $\mathcal{N} = \mathcal{H}^2(k)$, then we write D_n for $D_n^{{}_{\mathcal{H}}S_k}$.

Our first observaton is

Lemma 3.9.2. *D_n^C is a positive contraction for all $n \in \mathbb{N}$.*

Proof. Recall that 3.6.1 established that $H^2(k)$ has a basis $\{s_l\}$ dual to the basis $\{z^l\}$ with respect to the inner product (3.18). We begin with the case $\mathcal{H} = \mathbb{C}$, i.e. $\mathcal{H}^2(k) = H^2(k)$. Fix n and consider

$$< \left(\sum_{j=1}^{n} b_j S_k^j S_k^{*j} \right) s_q, s_r >= \sum_{j=1}^{n} b_j < s_{q-j}, s_{r-j} > . \tag{3.54}$$

If $q \neq r$ then the sum is 0. Otherwise, if $q = r$, then (3.54) is $\sum_{j=1}^{n} b_j a_{q-j}$. Interpreting $a_l = 0$ for $l < 0$ we have

$$\sum_{j=1}^{n} b_j a_{q-j} \leq \sum_{j=1}^{q} b_j a_{q-j} = a_q. \tag{3.55}$$

Hence we conclude

$$\sum_{j=1}^{n} b_j S_k^j S_k^{*j} \leq I. \tag{3.56}$$

In the case that $\mathcal{H} \neq \mathbb{C}$, it is clear that (3.56) holds with S_k replaced with $\mathcal{S}_k$.

Since $P_{\mathcal{N}}$ intertwines C and $\mathcal{S}_k$, and $P_{\mathcal{N}}$ is a partial isometry, we have $C = P_{\mathcal{N}} \mathcal{S}_k P_{\mathcal{N}}^*$. Hence $P_{\mathcal{N}} \mathcal{S}_k P_{\mathcal{N}}^* P_{\mathcal{N}} = C P_{\mathcal{N}} = P_{\mathcal{N}} \mathcal{S}_k$. Likewise $P_{\mathcal{N}}^* P_{\mathcal{N}} \mathcal{S}_k^* P_{\mathcal{N}}^* = \mathcal{S}_k^* P_{\mathcal{N}}^*$, upon taking adjoints. Hence we have

$$C^j C^{*j} = P_{\mathcal{N}} \mathcal{S}_k^j \mathcal{S}_k^{*j} P_{\mathcal{N}}^*. \tag{3.57}$$

Since $b_j \geq 0$ it then follows that (3.56) holds with C in place of S_k. The result then follows. $\square$

Lemma 3.9.3. *The limit*

$$\lim_{n \to \infty} D_n^C = D^C \tag{3.58}$$

exists in the strong operator topology.

Proof. Since $b_j \geq 0$, it is clear from inspection that $I \geq D_n^C \geq D_{n+1}^C \geq 0$ for all $n \in \mathbb{N}$. Hence for $m < n$ we have $||D_m^C - D_n^C|| \leq 1$. Together with several applications

of the Cauchy-Schwarz inequality this yields

$$\begin{aligned}\|D_m^C x - D_n^C x\|^4 &= \| < (D_m^C - D_n^C)x, (D_m^C - D_n^C)x > \|^2 \\ &\leq < (D_m^C - D_n^C)x, x >< (D_m^C - D_n^C)^2 x, (D_m^C - D_n^C)x > \\ &\leq < (D_m^C - D_n^C)x, x > \|(D_m^C - D_n^C)^2 x\| \, \|(D_m^C - D_n^C)x\| \\ &\leq \ (< D_m^C x, x > - < D_n^C x, x >)\|x\|^2 \end{aligned} \tag{3.59}$$

Since $I \geq D_m^C \geq D_{m+1}^C \geq 0$, $< D_n^C x, x >$ is a bounded decreasing sequence of numbers. The above calculation shows that $D_m^C x$ is a Cauchy sequence. Define $D^C x = \lim_{n \to \infty} D_n^C x$. Then an application of the Banach Steinhaus theorem will show that D^C is a bounded positive operator. □

Lemma 3.9.2 establishes that for each $n \in \mathbb{N}$ the operator D_n^C has a positive square root, which we denote by B_n^C.

Corollary 3.9.4. *The limit*

$$\lim_{n \to \infty} B_n^C = B^C \tag{3.60}$$

exists in the strong operator topology. Moreover

$$(B^C)^2 = D^C. \tag{3.61}$$

Proof. The proof of Lemma 3.9.3 applies *mutatis mutandis* to show that B^C is a bounded positive operator. The functional calculus for self adjoint operators then guarantees (3.61). □

Definition 3.9.5. For $n \in \mathbb{N}$, define $V_n : \mathcal{N} \to H^2(k) \otimes \mathcal{N}$ via

$$m \mapsto \sum_{l=0}^{n} s_l \otimes B_{(n-l)}^C (C^*)^l m, \tag{3.62}$$

$W_n : \mathcal{N} \to H^2(k) \otimes \mathcal{N}$ via

$$m \mapsto \sum_{l=0}^{n} s_l \otimes B^C (C^*)^l m, \tag{3.63}$$

and $\mathcal{W}_n : \mathcal{H}^2(k) \to H^2(k) \otimes \mathcal{H}^2(k)$ via

$$h \mapsto \sum_{l=0}^{n} s_l \otimes B_k^S S_k^{*l} h \tag{3.64}$$

Theorem 3.9.6. *With W_n as above, $W = \lim_{n\to\infty} W_n$ exists in the strong operator topology. Moreover, W is an isometry.*

Proof. We need to show that for fixed m, $Wm = \lim_{n\to\infty} W_n m$ exists, and $\|Wm\| = \|m\|$. An application of the Banach Steinhaus theorem then gives convergence in the strong operator topology. Towards this end we make the following observations.

Lemma 3.9.7. *$\mathcal{W} = \lim_{n\to\infty} \mathcal{W}_n$ exists in the strong operator topology and $\mathcal{W}$ is an isometry.*

Proof. Indeed, for fixed $h = \sum_{n=0}^{\infty} h^n \otimes s_n$, and $n > m$, we have

$$\begin{aligned} \|(\mathcal{W}_n - \mathcal{W}_m)h\|^2 &= \langle \sum_{l=m+1}^{n} a_l S_k^l D_k^S S_k^{*l} h, h\rangle \\ &= \sum_{l=m+1}^{n} a_l |h^l|^2. \end{aligned} \tag{3.65}$$

Hence $\mathcal{W}_n h$ is a Cauchy sequence in $H^2(k)\otimes\mathcal{H}^2(k)$. Moreover $\lim_{n\to\infty} \|\mathcal{W}_n h\| = \|h\|$, as the following calculation shows

$$\begin{aligned} \|\mathcal{W}_n h\| &= \langle \sum_{l=0}^{n} a_l S_k^l D_k^S S_k^{*l} h, h\rangle \\ &= \sum_{l=0}^{n} a_l |h^l|^2. \end{aligned} \tag{3.66}$$

So in fact $\mathcal{W}$ is an isometry ◻

Lemma 3.9.8. *For each $n \in \mathbb{N}$, V_n is an isometry.*

Proof. Again, calculating

$$
\begin{aligned}
\|V_n m\|^2 =& \langle \sum_{l=0}^{n} a_l C^l D_{n-l}^C (C^*)^l m, m\rangle && (3.67)\\
=& \langle \sum_{l=0}^{n} a_l C^l \left(\sum_{j=0}^{n-l} b_j C^j (C^*)^j \right) (C^*)^l m, m\rangle = \langle \sum_{l=0}^{n} \sum_{j=0}^{n-l} a_l b_j C^{l+j} (C^*)^{l+j} m, m\rangle \\
=& \langle \sum_{i=0}^{n} \sum_{n=0}^{i} a_n b_{i-n} C^i (C^*)^i m, m\rangle \\
=& \langle m, m\rangle.
\end{aligned}
$$

Hence V_n is an isometry. □

Now, we have the following

$$
\begin{aligned}
0 \geq \|W_n m\|^2 - \|V_n m\|^2 =& \langle \sum_{l=0}^{n} a_l \sum_{j=n+1-l}^{\infty} b_j C^{l+j} D^C (C^*)^{l+j} m, m\rangle && (3.68)\\
=& \langle \sum_{l=0}^{n} a_l \sum_{j=n+1-l}^{\infty} b_j \mathcal{S}_k^{l+j} P_{\mathcal{N}} (\mathcal{S}_k^*)^{l+j} m, m\rangle \\
\geq& \langle \sum_{l=0}^{n} a_l \sum_{j=n+1-l}^{\infty} b_j \mathcal{S}_k^{l+j} (\mathcal{S}_k^*)^{l+j} m, m\rangle \\
=& \|\mathcal{W}_n m\|^2 - \|m\|^2,
\end{aligned}
$$

which shows that $\lim_{n\to\infty} \|W_n m\| = \|m\|$, and hence $\{\|W_n m\|\}$ is Cauchy. The form of W_n then implies $\|W_j m - W_l m\|^2 = |\, \|W_j m\|^2 - \|W_l m\|^2 \,|$, and we see that $W_n m$ is a Cauchy sequence, thus establishing the theorem. □

Theorem 3.9.9. *Let $\mathcal{M}$ be a Hilbert space endowed with an operator C from $\mathcal{L}(\mathcal{M})$. Suppose that there is a surjective partial isometry $P_{\mathcal{N}} : \mathcal{H}^2(k) \twoheadrightarrow \mathcal{M}$ which intertwines the operators C and ${}_{\mathcal{H}}\mathcal{S}_k$. Let $\mathcal{K}$ denote the kernel of the partial isometry $P_{\mathcal{N}}$. Then there is a partial isometry $P_{\mathcal{K}} : H^2(k) \otimes \mathcal{K} \to \mathcal{K}$ which intertwines the operators ${}_{\mathcal{K}}\mathcal{S}_k$ and ${}_{\mathcal{H}}\mathcal{S}_k|_{\mathcal{K}}$.*

Proof. Theorem 3.9.6 shows that W maps $\mathcal{K}$ into $H^2(k) \otimes \mathcal{K}$ isometrically. Take $W* : H^2(k) \otimes \mathcal{K} \to \mathcal{K}$ as the partial isometry. Let $T =_{\mathcal{H}} \mathcal{S}_k|_{\mathcal{K}}$. Then we have the

following

$$
\begin{aligned}
{}_{\mathcal{H}}S_k^* W k &= \sum_{l=0}^{\infty} s_l \otimes D^T (T^*)^{l+1} k \\
&= \sum_{l=0}^{\infty} s_l \otimes D^T (T^*)^l T^* k \\
&= W T^* k.
\end{aligned}
\tag{3.69}
$$

Then upon taking adjoints we see that W^* is in fact an intertwining, as was to be shown. □

CHAPTER 4
HOMOLOGICAL MEANING

4.1 Introduction

In this chapter we show that the results of Chapter 3 establish the solution of several mapping problems in homological algebra. In particular we show that in the category in which we will work, projective objects exist. Moreover there are enough projectives in this category, in the sense that every object can be realized as the image of a projective. We go on to demonstrate that every object in the category then has a projective resolution. The discussion in Chapter 2 guarantees then the essential uniqueness of such a projective resolution. As a result it is then possible to define an **Ext** functor from this category to the category of abelian groups. In all that follows k denotes a fixed NP kernel, and $H^2(k)$ is defined as in Section 3.5.

4.2 The Category $\mathfrak{H}2$

In this section we define the category in which we will work. Recall from Section 2.1 that in Definition 2.1.1 a category provides a class of objects. Objects in the category $\mathfrak{H}2$ are pairs $(\mathcal{M}, T)$ where

1. T is a bounded operator on the separable Hilbert space $\mathcal{M}$,
2. there exists a separable Hilbert space $\mathcal{H}$ such that $\mathcal{M}$ is a subspace of $\mathcal{H} \otimes H^2(k)$, and
3. the orthogonal projection $P_{\mathcal{M}} : \mathcal{H} \otimes H^2(k) \to \mathcal{M}$ intertwines the operators T and $\mathcal{S}_k$.

Morphisms between objects $(\mathcal{M}, T)$ and $(\mathcal{N}, V)$ are bounded linear intertwinings. With these definitions is is routine to verify that $\mathfrak{H}2$ forms a category. It is convenient

to establish the following nomencalture. Given an object $(\mathcal{M}, T)$ from $\mathfrak{H}2$ and the space $\mathcal{H} \otimes H^2(k)$ in item 2 above, we say that $\mathcal{M}$ is a **-submodule* of $\mathcal{H} \otimes H^2(k)$.

It is in fact easy to verify that given two objects $(\mathcal{M}, T)$ and $(\mathcal{N}, V)$ from $\mathfrak{H}2$ the object $(\mathcal{M} \oplus \mathcal{N}, T \oplus V)$ is a product in the category. Indeed, there exist objects $(\mathcal{H}_1 \otimes H^2(k), \mathcal{S}_k^{(1)})$ and $(\mathcal{H}_2 \otimes H^2(k), \mathcal{S}_k^{(2)})$ of which $\mathcal{M}$, $\mathcal{N}$ are *-submodules, respectively. One then checks that $(\mathcal{M} \oplus \mathcal{N}, T \oplus V)$ is a *-submodule of $(\mathcal{H}_1 \oplus \mathcal{H}_2) \otimes H^2(k)$. Since addition of bounded intertwings produces bounded intertwinings, we have established

Theorem 4.2.1. *The category $\mathfrak{H}2$ is an additive category.*

Let $\mathcal{E}$ be the class of all sequences

$$\cdots \longrightarrow \mathcal{M}' \xrightarrow{\mu} {}'\mathcal{M} \xrightarrow{\mu} \mathcal{M}'' \longrightarrow \cdots \tag{4.1}$$

in which each object is an object from $\mathfrak{H}2$, each morphism is an intertwining partial isometry, and for μ' and μ successive morphisms in the sequence we have

$$\text{image}(\mu') = \text{kernel}(\mu).$$

Since we have not established the existence of (co)kernels in the category $\mathfrak{H}2$, this last requirement on the morphisms μ' and μ is established in the category of (separable) Hilbert spaces. We declare the elements of $\mathcal{E}$ to be the exact sequences in the category $\mathfrak{H}2$.

4.3 Projective Modules

In this section we demonstrate that projective objects exist in the category $\mathfrak{H}2$. We then show that every object in $\mathfrak{H}2$ is the image of a projective object. We begin with the

Theorem 4.3.1. *Let $\mathcal{H}$ be a complex separable Hilbert space. The object $(\mathcal{H} \otimes H^2(k), \mathcal{S}_k)$ is projective in the category $\mathfrak{H}2$.*

Proof. We have to solve the mapping problem described by (2.3):

$$\begin{tikzcd}
& \mathcal{H} \otimes H^2(k) \arrow[dl, dotted, "\psi"'] \arrow[d, "\phi"] \arrow[dr, "0"] & \\
E' \arrow[r, "\epsilon'"] & E \arrow[r, "\epsilon"] & E''
\end{tikzcd} \tag{4.2}$$

Since E' is in the category, there is a Hilbert space $\mathcal{M}$ such that E' is a *-submodule of $\mathcal{M} \otimes H^2(k)$. Note that ϕ maps $\mathcal{H} \otimes H^2(k)$ into the image of the partial isometry ϵ'. Hence if we can solve the mapping problem

$$\begin{tikzcd}
\mathcal{M} \otimes H^2(k) \arrow[d, "P_{E'}"] & \mathcal{H} \otimes H^2(k) \arrow[l, dotted, "\tilde{\psi}"'] \arrow[d, "\phi"] \arrow[dr, "0"] & \\
E' \arrow[r, "\epsilon'"] & E \arrow[r, "\epsilon"] & E''
\end{tikzcd} \tag{4.3}$$

where $P_{E'}$ denotes the orthogonal (intertwining) projection onto E', then taking the composition $P_{E'}\tilde{\psi} = \psi$ will solve the mapping problem (4.2). By applying Theorem 3.8.3 with $\epsilon' P_{E'}$ as P and ϕ as f, we see that there exists a bounded intertwining $\tilde{\psi}$ which solves the problem (4.3). In fact, Theorem 3.8.3 guarantees that $\|\tilde{\psi}\| \leq \|\phi\|$. □

As we stated in the beginning of this section, every object in $\mathfrak{H}2$ can be realized as the image of a projective object. Indeed, by fiat, objects in the category $\mathfrak{H}2$ are precisely those pairs $(\mathcal{M}, T)$ for which $\mathcal{M}$ could be realized as a *-submodule of some $\mathcal{H} \otimes H^2(k)$. As we have just seen, $\mathcal{H} \otimes H^2(k)$ is projective in $\mathfrak{H}2$, hence there are "enough" projectives in the category $\mathfrak{H}2$.

4.4 The Commutant Lifting Theorem

In this section we show that the Commutant Lifting Theorem appears in the category $\mathfrak{H}2$ as the solution to a mapping problem. Specifically, let $\mathcal{M}$ and $\mathcal{N}$ be two objects from $\mathfrak{H}2$, and *-submodules of $\mathcal{H}_1 \otimes H^2(k)$ and $\mathcal{H}_2 \otimes H^2(k)$, respectively. In this situation we have

Theorem 4.4.1. *For every morphism* $\mu : \mathcal{M} \to \mathcal{N}$ *there exists a morphism* $\tilde{\mu} : \mathcal{H}_1 \otimes H^2(k) \to \mathcal{H}_2 \otimes H^2(k)$ *making the following diagram commute*

$$\begin{tikzcd}
\mathcal{H}_1 \otimes H^2(k) \arrow[r, "P_{\mathcal{M}}"] \arrow[d, dotted, "\tilde{\mu}"'] & \mathcal{M} \arrow[d, "\mu"] \\
\mathcal{H}_2 \otimes H^2(k) \arrow[r, "P_{\mathcal{N}}"] & \mathcal{N}
\end{tikzcd} \tag{4.4}$$

Proof. The bottom row in the diagram (4.4) can be extended to end in 0. This extended bottom row is then an element of $\mathcal{E}$ the class of exact sequences. Since $\mathcal{H}_1 \otimes H^2(k)$ is a projective object in the category, there exists a morphism $\tilde{\mu}$ solving the diagram (4.2) with $\mu P_{\mathcal{M}}$ in place of ϕ from (4.2). □

4.5 The Existence of Resolutions

In this section we show that every object in the category $\mathfrak{H}2$ has a projective resolution. The key point in this demonstration is establishing that the kernel of the orthogonal projection from an object $\mathcal{H}_0 \otimes H^2(k)$ onto a *-submodule $\mathcal{M}$ is still in the category $\mathfrak{H}2$. Once the kernel is known to be in the category, we know that there is an object $\mathcal{H}_1 \otimes H^2(k)$ for which the kernel is a *-submodule. An induction then establishes the existence of the projective resolution of $\mathcal{M}$. The requirement that the kernel $\mathcal{K}$ of the orthogonal projection $P_{\mathcal{M}} : \mathcal{H}_0 \otimes H^2(k) \to \mathcal{M}$ is again in the category is given by Theorem 3.9.9. The beginning of the resolution, and the base case for the induction is represented as

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{K} \otimes H^2(k) \arrow[r] \arrow[d, "P_{\mathcal{K}}"] & \mathcal{H}_0 \otimes H^2(k) \arrow[r, "P_{\mathcal{M}}"] & \mathcal{M} \arrow[r] & 0 \\
& \mathcal{K} \arrow[ur] & & &
\end{tikzcd} \tag{4.5}$$

Since $\mathcal{K} \otimes H^2(k)$ is again a projective object it serves as the second element in the projective resolution. The next element in the projective resolution is derived by repeating the above process.

With the establishment that every object in $\mathfrak{H}2$ has a projective resolution we can there define an $\mathbf{Ext}$ functor from the category $\mathfrak{H}2$ to the category of abelian groups as outlined in Section 2.3.

CHAPTER 5
CONCLUSION

5.1 Summary

We have shown that given a Nevanlinna-Pick kernel k we are able to construct a Hilbert space $H^2(k)$ of functions for which k is a reproducing kernel. We have shown that multiplication by the polynomial z is a bounded operator, denoted S_k, on $H^2(k)$. Hence we are able to view $H^2(k)$ as a module over the algebra of multipliers of $H^2(k)$- i.e. the algebra consisting of those elements $f \in H^2(k)$ such that the map $g \mapsto fg$ is a bounded map. From this point of view it is natural to investigate the representations of this algebra of multipliers. Such an investigation is a study of the module homomorphisms, or bounded intertwinings. The condition that an intertwining be continuous can be seen as essential as $H^2(k)$ is endowed with a norm structure.

Our approach focused upon a particular class of modules, which turned out to be projective objects in our category. Namely, given a complex separable Hilbert space $\mathcal{M}$, the operator ${}_{\mathcal{M}}S_k = I \otimes S_k : \mathcal{M} \otimes H^2(k) \to \mathcal{M} \otimes H^2(k)$ is bounded. We showed that under the hypothesis of Theorem 3.8.3 we were able to establish the existence of a bounded intertwining F which solved the following mapping problem

$$\begin{tikzcd} & \mathcal{H} \otimes H^2(k) \arrow[dl, dotted, "F"'] \arrow[d, "f"] \arrow[dr, "0"] & \\ \mathcal{M} \otimes H^2(k) \arrow[r, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0 \end{tikzcd} \tag{5.1}$$

provided $P_{\mathcal{N}}$ is a partial isometry intertwining the operators ${}_{\mathcal{M}}S_k$ on $\mathcal{M} \otimes H^2(k)$, and C on $\mathcal{N}$.

As a result we were able to establish Corollaries 3.8.4 and 3.8.5. In Chapter 4 we defined the category $\mathfrak{H}2$ and demonstrated that Corollary 3.8.4 means that the objects of the form $(\mathcal{H} \otimes H^2(k), {}_{\mathcal{H}}S_k)$ in the category $\mathfrak{H}2$ are projective. We also showed in Chapter 4 that Corollary 3.8.5 is a generalization of the Commutant Lifting Theorem.

Additionally we established Theorem 3.9.9 in Chapter 3. This result established the existence of a bounded intertwining which in Chapter 4 we showed meant that objects in the category $\mathfrak{H}2$ had projective resolutions. The essential uniqueness of a projective resolution in an additive category relative to a class of exact sequences was established in Chapter 2. Thus an **Ext** functor from the category $\mathfrak{H}2$ to the category of Abelian groups is well defined.

5.2 The Horizon

We now briefly outline some directions for further research. With the existence of an **Ext** functor established, one would like to be able to calculate **Ext** groups. In particular, one would like to be able to demonstrate that there exist a Nevanlinna-Pick kernel k and objects $(\mathcal{H}, T)$, and $(\mathcal{K}, C)$ such that $\mathbf{Ext}^2(\mathcal{H}, \mathcal{K}) \neq 0$. It is worthwhile to point out that in the classical case when k is the Szegő kernel that this will *never* be the case. Indeed, in this case, every object $(\mathcal{H}, T)$ has a projective resolution of the form

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathcal{H}. \tag{5.2}$$

Whether this is true or not for the Dirichlet kernel is not known at this time.

Similarly, given a kernel k one would like to be able to make a calculation of the homological dimension for the ring of multipliers of $H^2(k)$. A less ambitious goal than developing the tools necessary to answer this question for all k would be simply to find conditions upon k which would imply the homological dimension was greater than $2, 3, \ldots$ ...

If it can be established that higher **Ext** groups do exist, one might then delve into the structure of the **Ext** groups themselves. These groups are in fact groups of bounded linear intertwinings, and hence carry more structure then just that of an Abelian group. In particular, they can be endowed with a topology, and therefore carry at least the structure of a topological vector space (TVS).

Perhaps most ambitious of all might be the problem of realizing a particular TVS as a particular **Ext** group in a given dimension. Of course all that can be said of such a program now is that it lies on the horizon.

Other directions for research exist as well, in particular the further development of the categorical foundations. Specifically, we believe it possible to show that the category $\mathfrak{H}2$ is in fact abelian. Once established, several standard homological results such as the Snake Lemma will follow. The significance of the category $\mathfrak{H}2$ being abelian is that homology groups may be defined directly from complexes in $\mathfrak{H}2$.

We believe these questions are both interesting and instructive. Answers will lead to new insights, and we believe these insights will be both useful and productive.

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BIOGRAPHICAL SKETCH

Robert Stephen Clancy was born in Madrid, Spain, on July 22, 1965. He was adopted at birth by his father and mother, Robert and Wanda Clancy. He grew up in Central Florida with his sister Christine five years his senior, where he graduated from Palm Bay Senior High School in 1983. He attended the University of Florida and received a Bachelor of Science degree in physics in 1989. He continued on at the University of Florida in the Department of Mathematics earning a Masters of Science degree in 1991. In October of 1993, Robert met his birth mother Mary (Bromaghim) Bostwick, with whom he now has a lasting bond. In August of 1996, he met Jennifer Lynne Airoldi, with whom he now endeavors to spend as much time as possible.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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